ON MATHEMATICAL MODELLING OF HEAT AND MOISTURE DISTRIBUTION IN POROUS MULTILAYER MEDIA

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Abstract. In the paper, the problem of diffusion of one substance through the pores of a porous multi layered material which may absorb and immobilize some of the diffusing substances with evolution or absorption of heat is studied. As an example we considered a round wood-block with two layers in the radial direction. We have two processes, the transfer of moisture and the transfer of heat. We can derive the system of two partial differential equations (PDEs), one expressing the rate of change of the concentration of water vapour in the air spaces and the other - the rate of change of temperature in every layer. The approximation of the corresponding initial boundary value problem of the system of PDEs is based on the conservative averaging method. This procedure allows reducing the 3-D axis-symmetrical transfer problem described by a system of PDEs to the initial value problem for a system of ordinary differential equations (ODEs) of the first order.

Keywords: heating and moistening processes, axis-symmetrical diffusion problem, conservative averaging method, special splines, analytic and numerical solution.

1. The mathematical model

The study of hydrodynamic flow and heat transfer through a porous media becomes much more interesting due to its vast applications [1-3]. Many mathematical models are developed for the analysis of such processes, for example, mathematical models of moisture movement in wood, when the wood is considered as porous media [6; 7]. In this paper we study the heat and moisture transfer processes in the porous multilayered media layer. In one layer this process is analysed and described in [1; 2].

The process of diffusion is considered in 3-D domain

$$\Omega = \left\{ (r, z, \phi) : 0 \le r \le R, 0 \le z \le L, 0 \le \phi \le 2\pi \right\}.$$

The domain Ω consists of multilayer medium in the r direction. We will consider the non-stationary axis-symmetrical problem of the linear diffusion theory for multilayered piece-wise homogenous materials of N layers in the sub-domain

$$\Omega_i = \left\{ (r, z, \phi) : r \in (r_{i-1}, r_i), z \in (0, L), \phi \in (0, 2\pi) \right\}, i = \overline{1, N},$$

where $H_i = r_i - r_{i-1}$ – the thickness of the layer $\Omega_i, r_0 = 0, r_N = R$.

We shall further assume the linear dependence on both temperature and the moisture content in every layer [1]

$$M_i = const + \sigma_i C_i - \omega_i T_i, \tag{1}$$

where $C_i(r, z, t)$ – the concentration of water vapour in the air spaces;
M_i – the amount of moisture absorbed by unit mass of fibre;
σ_i and ω_i – constants.

We can derive two equations, one expressing the rate of change of the concentration and the other – the rate of change of temperature. The water vapour diffusion PDEs in the cylindrical coordinates is in the following form

$$m\left(\frac{1}{r}\left(\frac{\partial}{\partial r}\left(D_{ir} r \frac{\partial C_i}{\partial r}\right)\right) + D_{iz}\frac{\partial^2 C_i}{\partial z^2}\right) = m\frac{\partial C_i}{\partial t} + (1-m)\rho_s \frac{\partial M_i}{\partial t} : \in [r_{i-1}, r_i], i = \overline{1, N}, t > 0, \tag{2}$$

where D_{ir}, D_{iz} – the diffusion coefficients for moisture in the air;
m – the fraction of the total volume of the material occupied by the air
(1-m) – the fraction of the porous material occupied by fibre of density
ρ_s, t is the time.

If $m = 1$, then the equation is diffusion equation for the concentration without fibres.

The heat diffusion PDEs can be rewritten in the following form:

$$c_i \rho_i \frac{\partial T_i}{\partial t} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(K_{ir} r \frac{\partial T_i}{\partial r}\right)\right) + K_{iz}\frac{\partial^2 T_i}{\partial z^2} + \rho_i \frac{\partial M_i}{\partial t};\ r \in [r_{i-1}, r_i],\ i = \overline{1, N},\ t > 0, \tag{3}$$

where c_i – the specific heat of the fibres;
K_{ir}, K_{iz}, ρ_i – the heat conductivities and the densities of the porous material;
q_i – the heat evolved when the water vapour is absorbed by the fibres.

We assume that all coefficients in the PDEs are assumed as constants and are independent of the moisture concentration and temperature.

By eliminating M_i from (1), we get the system of two PDEs

$$\begin{cases} D_{ir}^{T} \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}\left(r \dfrac{\partial T_i}{\partial r}\right)\right) + D_{iz}^{T} \dfrac{\partial^2 T_i}{\partial z^2} = \dfrac{\partial T_i}{\partial t} - v_i \dfrac{\partial C_i}{\partial t} \\ D_{ir}^{C} \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}\left(r \dfrac{\partial C_i}{\partial r}\right)\right) + D_{iz}^{C} \dfrac{\partial^2 C_i}{\partial z^2} = \dfrac{\partial C_i}{\partial t} - \lambda_i \dfrac{\partial T_i}{\partial t} \end{cases}, \tag{4}$$

where $i = \overline{1, N}$, $D_{ir}^{T} = \dfrac{K_{ir}}{\rho_i (c_i + q_i \omega_i)}$, $D_{ir}^{C} = \dfrac{D_{ir}\, m}{m + (1-m)\rho_s \sigma_i}$, $D_{iz}^{T} = \dfrac{K_{iz}}{\rho_i (c_i + q_i \omega_i)}$,

$D_{iz}^{C} = \dfrac{D_{iz}\, m}{m + (1-m)\rho_s \sigma_i}$, $v_i = \dfrac{q_i \sigma_i}{c_i + q_i \omega_i}$, $\lambda_i = \dfrac{(1-m)\omega_i\, \rho_s}{m + (1-m)\rho_s \sigma_i}$, $\lambda_i v_i < 1$.

For the initial condition for $t = 0$ we define the following values $T_i(r, z, 0) = T_0$, $C_i(r, z, 0) = C_0$, $i = \overline{1, N}$, where T_0, C_0 are known constants. We use the following boundary and continuous conditions:

$$\begin{cases} \dfrac{\partial Q_1(0, z, t)}{\partial r} = 0,\ Q_N(R, z, t) = Q_{ar},\ Q_N(r, L, t) = Q_{az},\ \dfrac{\partial Q_i(r, 0, t)}{\partial z} = 0,\ i = \overline{1, N}, \\ Q_i(r_i, z, t) = Q_{i+1}(r_i, z, t),\ D_{ir}^{Q} \dfrac{\partial Q_i(r_i, z, t)}{\partial r} = D_{i+1,r}^{Q} \dfrac{\partial Q_{i+1}(r_i, z, t)}{\partial r},\ i = \overline{1, N-1}, \end{cases} \tag{5}$$

where $Q_i = Q_i(r_i, z, t)$, $Q = (T, C)$ – the solutions in every layer;
T_{ar}, T_{az}, C_{ar}, C_{az} – the given temperature and concentration on the boundary.

2. The conservative averaging method in z- direction

We consider the conservative averaging method (CAM) of the special integral splines with hyperbolic trigonometric functions for solving the 3-D initial boundary-value problem in the z-direction [4; 5]. This procedure allows reduce the 2-D problem in r,z-directions to a 1-D problem in the r-direction. Using CAM in the z-direction with the parameters a_{iz} we have

$$Q_i(r, z, t) = Q_{iv}(r, t) + m_i^{Q}(r, t)\frac{0.5L\sinh(a_{iz}(z - 0.5L))}{\sinh(0.5 a_{iz} L)} + e_i^{Q}(r, t)\left(\frac{\cosh(a_{iz}(z - 0.5L)) - A_{i0}}{8\sinh^2(a_{iz} L/4)}\right),$$

where $Q_{iv}(r, t) = L^{-1}\int_0^L Q_i(r, z, t)\,dz$, $Q_i = (T_i; C_i)$, $A_{i0} = \dfrac{\sinh(a_{iz} L/2)}{a_{iz} L/2}$.

The parameters a_{iz} are possible to be chosen for minimizing the maximal error. We can see that the parameters $a_{iz} > 0$ tend to zero, then the limit is the integral parabolic spline (A. Buikis [4]) because:

$$A_{i0} \to 1:\ Q_i(r,z,t) = Q_{iv}(r,t) + m_i^Q (z-0.5L) + e_i^Q\left(\frac{z-0.5L}{L^2} - \frac{1}{12}\right).$$

The unknown functions $m_i = m_i^Q(r,t),\ e_i = e_i^Q(r,t)$ can be determined from conditions (5):

1) for $z = 0$ $m_i d_i - e_i k_i = 0$, $m_i = e_i p_i$, $p_i = k_i / d_i$,

2) for $z = L$, $Q_{az} = Q_{iv} + m_i L/2 + e_i b_i$, $e_i = (Q_{az} - Q_{iv})/g0_i$,

where $d_i = 0.5La_{iz}\coth(0.5\,a_{iz}L)$,

$k_i = 0.25a_{iz}\coth(0.25\,a_{iz}L)$,

$b_i = (\cosh(a_{iz}L/2) - A_{io}) / (8\sinh^2(a_{iz}L/4))$,

$g0_i = b_i + 0.5L\,p_i$.

Now the 1-D initial-value problem (4) is in the following form

$$\begin{cases} D_{ir}^T \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T_{iv}}{\partial r}\right)\right) + D_{iz}^T a0_i^2 (T_{az} - T_{iv}) = \dfrac{\partial T_{iv}}{\partial t} - v_i \dfrac{\partial C_{iv}}{\partial t}, \\ D_{ir}^C \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}\left(r\dfrac{\partial C_{iv}}{\partial r}\right)\right) + D_{iz}^C a0_i^2 (C_{az} - C_{iv}) = \dfrac{\partial C_{iv}}{\partial t} - \lambda_i \dfrac{\partial T_{iv}}{\partial t}, \\ \dfrac{\partial T_{iv}(0,t)}{\partial r} = 0,\ \dfrac{\partial C_{iv}(0,t)}{\partial r} = 0,\ T_{Nv}(R,t) = T_{ar},\ C_{Nv}(R,t) = C_{ar}, \\ T_{iv}(r_i,t) = T_{i+1,v}(r_i,t),\ D_{ir}^T \dfrac{\partial T_{iv}(r_i,t)}{\partial r} = D_{i+1,r}^T \dfrac{\partial T_{i+1,v}(r_i,t)}{\partial r},\ i = \overline{1, N-1}, \\ C_{iv}(r_i,t) = C_{i+1,v}(r_i,t),\ D_{ir}^C \dfrac{\partial C_{iv}(r_i,t)}{\partial r} = D_{i+1,r}^C \dfrac{\partial C_{i+1,v}(r_i,t)}{\partial r},\ i = \overline{1, N-1}, \\ T_{iv}(r_i,0) = T_0,\ C_{iv}(r_i,0) = C_0,\ i = \overline{1,N}. \end{cases} \tag{6}$$

where $a0_i^2 = 2k_i / (L\,g0_i)$.

3. CAM in r- direction for N-layers

Using the averaged method with the parameters a_{ir} we have

$$Q_{iv}(r,t) = Q_{ivv}(t) + m_{ir}^Q(t)\left(\frac{0.5H_i\bar{r}_i (a_{ir}^Q)^2 \sinh(a_{ir}^Q (r-\bar{r}_i))}{\sinh(0.5a_{ir}^Q H_i)(d_i - 1)} - 1\right) + e_{ir}^Q(t)\left(\frac{\cosh(a_{ir}^Q(r-\bar{r}_i)) - A_{i1}^Q}{8\sinh^2(a_{ir}^Q H_i/4)}\right),$$

where $Q_{ivv}(t) = (H_i\bar{r}_i)^{-1}\int_{r_i-1}^{r_i} rQ_{iv}(r,t)dr$,

$\bar{r}_i = (r_{i-1} + r_i)/2$, $\bar{r}_i = (r_{i-1} + r_i)/2$, $r \in [r_{i-1}, r_i]$,

$A_{i1}^Q = \dfrac{\sinh(a_{ir}^Q H_i/2)}{a_{ir}^Q H_i/2}$, $d_i^Q = 0.5H_i a_{ir}^Q \coth(0.5a_{ir}^Q H_i), i = \overline{1,N}$.

We can use the following values of the parameters:

$$a_{ir}^T = a0_i\sqrt{\frac{D_{iz}^T}{D_{ir}^T}},\ a_{ir}^C = a0_i\sqrt{\frac{D_{iz}^C}{D_{ir}^C}}.$$

We can see, if the parameters $a_{ir}^Q > 0$ tend to zero, then we obtain the limit as an integral parabolic spline:

$$Q_{iv}(r,t)=Q_{ivv}+m_{ir}^{Q}\left(\frac{12\bar{r}_i}{H_i^2}(r-\bar{r}_i)-1\right)+e_{ir}^{Q}\left(\frac{(r-\bar{r}_i)^2}{H_i^2}-\frac{1}{12}\right).$$

From the boundary conditions (5) follows the system of 2N-linear equations for determining the unknown functions $m_i=m_{ir}^{Q}(t)$, $e_i=e_{ir}^{Q}(t)$:

1) for $r=0$ $m_1 d_{1r}^{Q}-e_1 k_{1r}^{Q}=0$,

2) for $r=R$, $Q_{ar}=Q_{Nvv}+m_N b_{Nm}^{Q}+e_N b_{Ne}^{Q}$,

3) for $r=r_i$, $Q_{ivv}+m_i b_{im}^{Q}+e_i b_{ie}^{Q}=Q_{i+1,vv}-m_{i+1}\left(b_{i+1,m}^{Q}+2\right)+e_{i+1}b_{i+1,e}^{Q}$,

$D_{ir}^{Q}m_i d_{ir}^{Q}+e_i k_{ir}^{Q}=D_{i+1,r}^{Q}m_{i+1,}d_{i+1,r}^{Q}-e_{i+1}k_{i+1,r}^{Q}$, $\overline{i=1,N-1}$,

where $d_{ir}^{Q}=\dfrac{d_i^{Q}\bar{r}_i\left(a_{ir}^{Q}\right)^2}{d_i^{Q}-1}$,

$k_{ir}^{Q}=0.25a_{ir}^{Q}\coth(0.25\,a_{ir}^{Q}H_i)$,

$b_{ir}^{Q}=\left(\cosh(a_{ir}^{Q}H_i/2)-A_{i1}^{Q}\right)/\left(8\sinh^2(a_{ir}^{Q}H_i/4)\right)$,

$b_{im}^{Q}=\dfrac{0.5H_i\bar{r}_i\left(a_{ir}^{Q}\right)^2}{d_i^{Q}-1}-1$.

Now the 1D initial- value problem (6) is in the form of the following ODEs system:

$$\begin{cases} D_{ir}^{T}\left(m_{ir}^{T}(t)d_{ir}^{T}/\bar{r}_i+2e_{ir}^{T}(t)k_{ir}^{T}\right)+D_{iz}^{T}\left(a0_i\right)^2\left(T_{az}-T_{ivv}(t)\right)=\dfrac{\partial T_{ivv}(t)}{\partial t}-v_i\dfrac{\partial C_{ivv}(t)}{\partial t}, \\ D_{ir}^{C}\left(m_{ir}^{C}(t)d_{ir}^{C}/\bar{r}_i+2e_{ir}^{C}(t)k_{ir}^{C}\right)+D_{iz}^{C}\left(a0_i\right)^2\left(C_{az}-C_{ivv}(t)\right)=\dfrac{\partial C_{ivv}(t)}{\partial t}-v_i\dfrac{\partial T_{ivv}(t)}{\partial t}, \\ T_{ivv}(0)=T_0,\, C_{ivv}(0)=C_0,\, i=\overline{1,N}. \end{cases} \tag{7}$$

4. CAM for two layers

For $N=2$ we have the following equations ($m_i=m_{ir}^{Q}, e_i=e_{ir}^{Q}, k_i=k_{ir}^{Q}, b_{im}=b_{im}^{Q}, b_{ie}=b_{ie}^{Q}$, $d_{ir}=d_{ir}^{Q}$):

$$m_1=p_1e_1,\, p_1=k_1/d_{1r},\, Q_{ar}=Q_{2vv}+m_2b_{2m}+e_2b_{2e},$$

$$Q_{1vv}+m_1b_{1m}+e_1b_{1e}=Q_{2vv}-m_2\left(b_{2m}+2\right)+e_2b_{2e},\; m_1d_{1r}+e_1k_1=D_{1,2}^{Q}(m_2d_{2r}-e_2k_2),$$

where $D_{1,2}^{Q}=D_{2,r}^{Q}/D_{1,r}^{Q}$.

Then

$$e_i=e_{i1}^{Q}Q_{1vv}+e_{i2}^{Q}Q_{2vv}+e_{i0}^{Q}Q_{ar},\; m_i=m_{i1}^{Q}Q_{1vv}+m_{i2}^{Q}Q_{2vv}+m_{i0}^{Q}Q_{ar},\; i=1;2,$$

where $e_{11}^{Q}=-(b_2^{Q}+1)/g_2^{Q}$, $e_{12}^{Q}=1/g_2^{Q}$, $e_{10}^{Q}=b_2^{Q}/g_2^{Q}$, $m_{11}^{Q}=e_{11}^{Q}p_1$,

$m_{12}^{Q}=e_{12}^{Q}p_1, m_{10}^{Q}=e_{10}^{Q}p_1$, $m_{21}^{Q}=-0.5\left(1+e_{11}^{Q}g_1^{Q}\right)/\left(1+b_{2m}\right)$,

$m_{22}^{Q}=-0.5e_{12}^{Q}g_1^{Q}/\left(1+b_{2m}\right)$, $m_{20}^{Q}=0.5\left(1-e_{10}^{Q}g_1^{Q}\right)/\left(1+b_{2m}\right)$,

$e_{21}^{Q}=m_{21}^{Q}/p_2-e_{11}^{Q}\cdot b_3^{Q}$, $e_{22}^{Q}=m_{22}^{Q}/p_2-e_{12}^{Q}\cdot b_3^{Q}$, $e_{20}^{Q}=m_{20}^{Q}/p_2-e_{10}^{Q}\cdot b_3^{Q}$,

$p_2=k_2/d_{2r}$, $g_2^{Q}=g_1^{Q}\left(1+b_2^{Q}\right)+b_3^{Q}b_{2e}$, $g_1^{Q}=p_1b_{1m}+b_{1e}$, $b_2^{Q}=0.5b_1^{Q}/\left(b_{2m}+1\right)$,

$b_1^Q = -b_{2m} - 2 + b_{2e} / p_2$, $b_3^Q = 2k_1 / k_2 D_{1,2}^Q$.

For $N = 2$ we have the following ODEs initial- value problem:

$$\begin{cases} \dot{T}_{ivv}(t) = \dfrac{1}{(1-\lambda_i v_i)}\left(b_{i1}^T T_{1vv}(t) + b_{i2}^T T_{2vv}(t) + T_{i0} + v_i\left(b_{i1}^C C_{1vv}(t) + b_{i2}^C C_{2vv}(t) + C_{i0}\right)\right), \\ \dot{C}_{ivv}(t) = \dfrac{1}{(1-\lambda_i v_i)}\left(b_{i1}^C C_{1vv}(t) + b_{i2}^C C_{2vv}(t) + C_{i0} + \lambda_i\left(b_{i1}^T T_{1vv}(t) + b_{i2}^T T_{2vv}(t) + T_{i0}\right)\right), \\ T_{ivv}(0) = T_0,\ C_{ivv}(0) = C_0, \end{cases} \tag{8}$$

where $b_{ik}^Q = D_{ir}^Q\left(g_{i3} m_{ik}^Q + g_{i4} e_{ik}^Q\right) - \delta_{i,k} D_{iz}^Q (a0_i)^2$, $k = 1;2$, $g_{i3} = d_{ir} / \bar{r}_i$, $g_{i4} = 2(k_{ir} / H_i)$,

$Q_{i0} = D_{ir}^Q\left(g_{i3} m_{i0}^Q + g_{i4} e_{i0}^Q\right) Q_{ar} + D_{iz}^Q (a0_i)^2 Q_{az}$, $i = 1;2$, $Q = (T, C)$,

$\delta_{i,k}$ – Kronecker symbol.

We rewrite the system of ODEs (8) in the following vector form:

$$\dot{W}(t) + AW(t) = F, W(0) = W_0$$

where $W(t), W_0, F$ – the 4-order vector-column with elements

$\left(T_{1vv}(t), T_{2vv}(t), C_{1vv}(t), C_{2vv}(t)\right), \left(T_0, T_0, C_0, C_0\right)$,

$F_1 = G_1(T_{10} + v_1 C_{10})$

$F_3 = G_1(\lambda_1 T_{10} + C_{10}), F_4 = G_2(\lambda_2 T_{20} + C_{20})$,

$G_1 = \dfrac{1}{(1-\lambda_1 v_1)}, G_2 = \dfrac{1}{(1-\lambda_2 v_2)}$.

A is the 4-order matrix

$$A = \begin{pmatrix} G_1 b_{11}^T & G_1 b_{12}^T & G_1 v_1 b_{11}^C & G_1 v_1 b_{12}^C \\ G_2 b_{21}^T & G_2 b_{22}^T & G_2 v_2 b_{21}^C & G_2 v_2 b_{22}^C \\ G_1 \lambda_1 b_{11}^T & G_1 \lambda_1 b_{12}^T & G_1 b_{11}^C & G_1 b_{12}^C \\ G_2 \lambda_2 b_{21}^T & G_2 \lambda_2 b_{22}^T & G_2 b_{21}^C & G_2 b_{22}^C \end{pmatrix}.$$

The averaged solution is $W(t) = \exp(-At)W_0 + \left(E - \exp(-At)\right)A^{-1}F$ and

$$Q_{iv}(r,t) = Q_{ivv}(t) + m_{ir}^Q(t)\left(\frac{0.5 H_i \bar{r}_i (a_{ir})^2 \sinh(a_{ir}(r - \bar{r}_i))}{\sinh(0.5 a_{ir} H_i)(d_i - 1)} - 1\right) + e_{ir}^Q(t)\left(\frac{\cosh(a_{ir}(r - \bar{r}_i)) - A_{i1}}{8\sinh^2(a_{ir} H_i / 4)}\right),$$

where $i = 1;2, Q = (T, C)$.

5. The conservative averaging method for model equations

For modelling the averaging method in 2 layers and estimate the parameter $a_i = a_{ir}^T$ we consider the stationary 1-D boundary-value problem (6) only for temperature with $T = T(r)$, $v = 0$, $T_{az} = 0$ $T_{ar} = T_0$, $D_{iz}^T = 1$, $D_{ir}^T = D_i$, $i = 1;2$ in the following form:

$$\begin{cases} D_1 r^{-1}(rT_1'(r))' - a0_1^2 T_1(r) = F_1,\ r \in [0, H_1],\ T_1'(0) = 0, \\ D_2 r^{-1}(rT_2'(r))' D - a0_2^2 T_2(r) = F_2,\ r \in [H_1, R],\ T_2(R) = T_{0,}, \\ T_1(H_1) = T_2(H_1),\ D_1 T_1'(H_1) = D_2 T_2'(H_1),\ 0 < H_1 < R, \end{cases} \tag{10}$$

where $T_0, F_i, a0_i > 0, i = 1;2$ – given constants.

The analytical solution is

$$T_1(r) = C_1 I_0(a_1 r) - f_1,\ T_2(r) = C_2 I_0(a_2 r) + C_3 K_0(a_2 r) - f_2,$$

where I_0, K_0 – the modified Bessel functions,

$f_i = F_i / a0_i^2,\ a_i = a0_i / \sqrt{D_i},\ i = 1;2$.

The unknown constants are obtained from boundary conditions (10) $(I_0' = I_1,\ K_0' = -K_1)$:

$$C_2 = \frac{T_0 - C_3 K_0(a_2 r)}{I_0(a_2 r)},\ C_3 = \frac{b_1 - f_a}{b_3},\ b_1 = \frac{b_0 T_0}{I_0(a_2 r)},$$

$$b_0 = D_{2,1} a_2 I_0(a_1 H_1) I_1(a_2 H_1) - a_1 I_1(a_1 H_1) I_0(a_2 H_1),\ f_a = a_1 I_1(a_1 H_1)(f_1 - f_2),$$

$$C_1 = D_{2,1}\left(C_2 a_2 I_1(a_2 H_1) - C_3 a_2 K_1(a_2 H_1)\right) / \left(a_1 I_1(a_2 H_1)\right),\ D_{2,1} = D_2 / D_1,$$

$$b_3 = b_0 \frac{K_0(a_2 R)}{I_0(a_2 R)} + b_2 = D_{2,1} a_2 I_0(a_1 H_1) K_1(a_2 H_1) + a_1 I_1(a_1 H_1) K_0(a_2 H_1).$$

With the hyperbolic averaging method in two layers we have:

$$T_i(r) = T_{iv} + m_i\left(\frac{0.5 H_i \bar{r}_i a_i^2 \sinh(a_i(r - \bar{r}_i))}{\sinh(0.5 a_{ir}^Q H_i)(d_i - 1)} - 1\right) + e_i\left(\frac{\cosh(a_i(r - \bar{r}_i)) - A_i}{8\sinh^2(a_i H_i / 4)}\right)$$

where $i = 1;2,\ \bar{r}_1 = 0.5 H_1,\ \bar{r}_2 = 0.5(H_1 + R),\ A_i = \left(\frac{\sinh(a_i H_i / 2)}{a_i H_i / 2}\right),$

$d_i = 0.5 H_i a_i \coth(0.5 a_i H_i)$.

The unknown constants T_{iv} can be determined from the first stationary equations (8):

$$T_{1v} = \left((F_1 - T_{10}) b_{22}^T - (F_2 - T_{20}) b_{12}^T\right) / \det,\ T_{2v} = \left(-(F_1 - T_{10}) b_{21}^T - (F_2 - T_{20}) b_{11}^T\right) / \det,$$

$$\det = b_{22}^T b_{11}^T - b_{21}^T b_{12}^T.$$

Using computer program MATLAB for $D_1 = 10^{-2}$, $D_2 = 10^{-1}$, $F_1 = -2$, $F_2 = 0$, $a_1 = 20$, $a0_1 = 2$, $a0_2 = 6$, $a_2 = 18.9737$, $T_0 = 4$, $R = 3$, $H_1 = 1.8$ we have the following maximal error:

1) 2.6275 for parabolic spline $(a_1 = a_2 = 10^{-4})$,
2) 0.0614 for hyperbolic spline (the minimal error 0.0602 is for $a_1 = 20$, $a_2 = 19.4737$).

6. Some numerical results

The results of the calculations are obtained by MATLAB. We use the discrete values $r_j, t_n = n\tau$, $n = \overline{0, N_t}$, $N_t \tau = t_f$. We consider the drying process in the wood-block with $L = 0.5m, R = 0.3m$, $H_1 = 0.21m, H_2 = 0.09m, T = const$ in two layers by:

$t_f = 1000 - 10000$, $C_0 = 1, C_{0r} = C_{0z} = 0$, $D_{1z} = 10^{-5}\ \mathrm{m \cdot s^{-2}}, D_{2z} = 10^{-4}\ \mathrm{m \cdot s^{-2}}$, $D_{1r} = 10^{-7}\ \mathrm{m \cdot s^{-2}}$, $D_{2r} = 10^{-6}\ \mathrm{m \cdot s^{-2}}$, $N_t = 1000, N_r = 30, N_z = 10$, $m = \lambda = 0$, $\nu = 1$, $a_{1z} = 10, a_{2z} = 1$.

The results of calculation are represented in Table 1 and Fig. 1-4, where MC_v, MC are the maximal values of the concentration C_v, C and mC_{1vv}, mC_{2vv} are the minimal value of averaged concentration C_{1vv}, C_{2vv} in every layer, obtained in the time moment t_f.

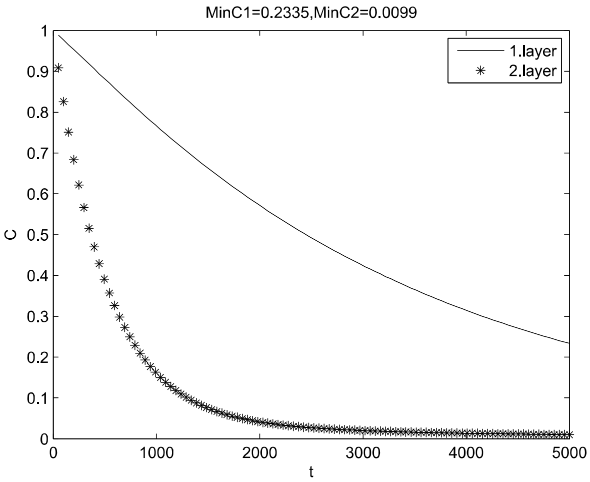

Fig. 1. **Averaged values C_{1vv}, C_{2vv} depending on t**

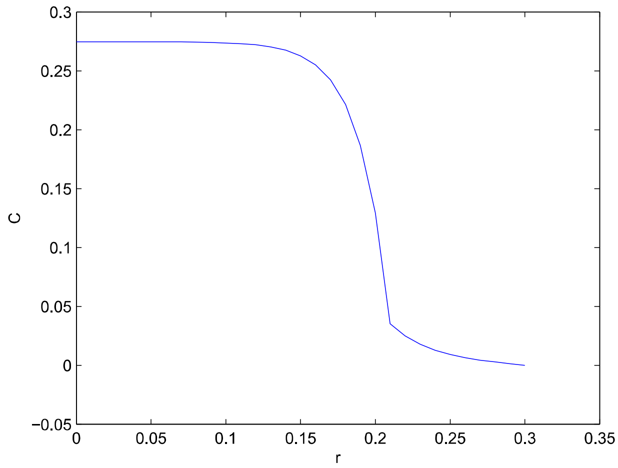

Fig. 2. **Averaged values C_{vv} depending on r for t = 5000 s**

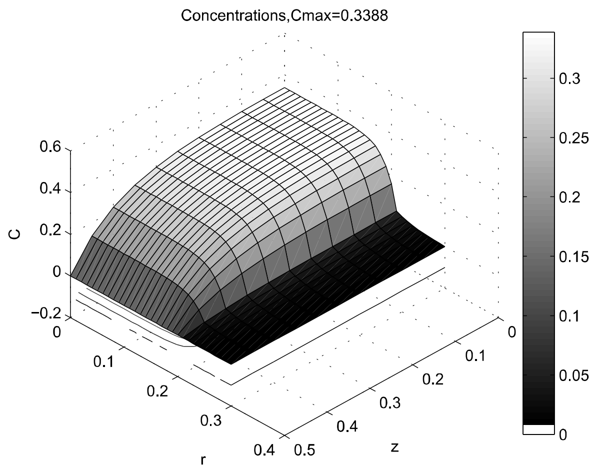

Fig. 3. **Concentration C depending on (r, z) for t = 5000 s**

Fig. 4. **Wood-block**

For t_f = 30000 the values are equal to 10^{-5}. For increasing λ the concentration of water vapour in the air spaces can be increased (see Table 1).

Table 1

The values of mC_{1vv}, mC_{2vv}, MC_v, MC

t_f	mC_{1vv}	mC_{2vv}	MC_v	MC	λ	v
1000	0.776	0.160	0.868	0.909	0	1
2000	0.571	0.041	0.668	0.824	–	–
3000	0.424	0.020	0.499	0.615	–	–
4000	0.315	0.014	0.370	0.457	–	–
5000	0.233	0.010	0.275	0.339	–	–
6000	0.173	0.007	0.204	0.251	–	–
7000	0.128	0.005	0.151	0.186	–	–
8000	0.095	0.004	0.112	0.138	–	–
9000	0.071	0.003	0.083	0.103	–	–
10000	0.052	0.002	0.061	0.076	–	–
20000	0.003	0.000	0.003	0.004	–	–
1000	0.843	0.177	0.955	1.170	0.1	1
5000	0.257	0.011	0.302	0.373	0.1	1

7. Conclusions

For investigation the heat and moisture transfer in porous multi-layer 3-D domain the system of two PDEs for determination the concentration C of water vapour in the air spaces and the temperature T is considered.

The approximation of the corresponding initial boundary value problem of the system of PDEs is based on the conservative averaging method (CAM), where brand-new hyperbolic type splines are used.

For these splines the best parameter for the minimal error is calculated. The problem of the system of 3-D PDEs with constant coefficients is approximated of the initial value problem of a system of ODEs of the first order.

Such a procedure allows us to obtain an analytical solution with a simple engineering algorithm for mass transfer equations for different substances in layered domain. It is possible modelling round and angular wood-blocks and gives some new physical conclusions about the drying and moister processes in these blocks, for example, it is obtained that in the drying process with increasing the parameter λ the concentration of water vapour in the air spaces is increased.

It was obtained that the wood drying process in the outer layer is faster than in the inner layer, which corresponds to the task approach, the mathematical model initial conditions and boundary conditions and the drying process, the actual carrying out.

Therefore, a mathematical model is appropriate for capturing and modelling the distribution of the drying process during the layers in accordance with the thickness of the layers and physical parameters.

The developed model allows calculate the time after which the moisture content of the wood will be reduced by a certain amount.

This allows the use of a mathematical model of the drying process for forecasting the time – calculate the required wood drying time after which the moisture content will drop to the necessary limit (norm).

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